# One-loop $N$-point equivalence among negative-dimensional, Mellin-Barnes and Feynman parametrization approaches to Feynman integrals

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We show that at one-loop order, negative-dimensional, Mellin-Barnes' (MB) and Feynman parametrization (FP) approaches to Feynman loop integrals calculations are equivalent. Starting with a generating functional, for two and then for $N$-point scalar integrals we show how to reobtain MB results, using negative-dimensional and FP techniques. The $N-$point result is valid for different masses, arbitrary exponents of propagators and dimension.

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## I. INTRODUCTION

The amazing comparison[1] between experimental determination and theoretical prediction of the anomalous magnetic momentum of the electron, is the greatest motivation - in our opinion - to study and develop techniques that allow the precise calculation of higher order Feynman loop integrals. Recently there are also interest in studying process like[2] $e^+e^- \to 3$ jets and $e^+e^- \to 4$ jets , so loop integrals with five and six external legs must be known.

Physicists' battle against the tricky Feynman loop integrals is fought in many fronts. We can cite some of them: integration by parts method[3] seems to be the most powerful one, since one can in most cases reduce the number of loops, e.g., the scalar massless two-loop master can be rewritten as a sum of two simpler integrals: a two-loop self-energy with an insertion[4, 5] plus the square of a one-loop self-energy. This is a very simple example − where one must deal with a greater number of simpler integrals with powers of propagators shifted − of a powerful technique, see for instance a 5-loop calculation in [6].

Integration by parts is used also associated with another methods. In fact, one can not evaluate a Feynman loop integral using the above mentioned technique alone. It simplifies the original diagram but does not solve it. In order to carry the integration out Gehrmann and Remiddi [7] did use the differential equation method, introduced by Kotikov[8], and solved a large class of difficult of problems. Glover and collaborators completed the study of the whole class $2 \to 2$ of two-loop scattering[9]. Also, Gegenbauer polynomial method has been used in order to study complicated process[10].

Other methods that make use of decomposition of complicated integrals, like the one-loop pentagon[11], were developed as well as string inspired ones[12]. See also [13–16] for other important approaches, a very powerful numerical technique on [17] and for a review on the progress of loop calculations[18].

Mellin-Barnes(MB) and negative dimensional integration method (NDIM) are two other interesting and powerful techniques to tackle such Feynman integrals. Mellin-Barnes approach relies on the relation,

$$\left(\sum_{i=1}^{n} z_i\right)^{-B} = \frac{1}{z_1^B (2\pi i)^{n-1}\Gamma(B)} \int_{-i\infty}^{i\infty} \Gamma(w_1+w_2+...+w_{n-1}+B) \prod_{i=1}^{n-1}\left[dw_i \left(\frac{z_{i+1}}{z_1}\right)^{w_i} \Gamma(-w_i)\right]. \qquad (1)$$

in other words, we rewrite each propagator as a Mellin transform. However, these parametric integrals are not difficult to solve - as it happen in the Feynman parametrization approach where the integrals become more and more complicated - because one can apply Cauchy theorem and two Barnes' lemmas[19]. The MB approach is being greatly used by Tausk[20], Smirnov[21], Davydychev[22, 23] and co-workers in order to tackle two and three-loop integrals. The results are always expressed as generalized hypergeometric functions which depend on adimensional ratios of momenta and/or masses, space-time dimension $D$ and exponents of propagators.

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On the other hand NDIM is a technique whereby it is not necessary to introduce parametric integrals, the Feynman integral is the result between the comparison of two calculations: a gaussian-like integral (the generating functional of negative-dimensional integrals) and a Taylor expansion of the generating functional[24]. It is worth mentioning that in the NDIM context performing the calculation for a particular set of exponents of propagators present the same difficulties than perform the same task for arbitrary values of them. The results are, just like in the MB approach, given in terms of generalized hypergeometric functions which depend on the same quantities mentioned above.

One can then rightfully ask: is there any connection between these two approaches? The answer is yes. The propose of our paper is to show that they are equivalent, at least at one-loop order. In fact, one could argument that the results must be the same if one correctly applies both methods, we will show this explicitly. However, it can be useful and interesting to study which of them is more powerful when the number of external legs increase. In one hand NDIM demands computer facilities in order to solve a large number of systems and browse the big number of results; on the other hand MB does not require computers but the integrals must be calculated one by one. Another point to observe is that NDIM relies on grassmannian integrals and MB on Mellin transform, i.e., apparently disconnect (as far as we know) subjects. Also, showing the equivalence between them and knowing the routes which take to NDIM or to MB one could build, for instance, a technique like NDIM in order to tackle problems in finite temperature field theory, like calculations of heat-kernel which can be dealt with using Mellin integrals.

The outline for our paper is as follows: in section II we present a step-by-step calculation of 2-point scalar integral starting with NDIM approach and arriving at an expression originally obtained by Davydychev[23] using MB approach, then we repeat the same process using FP. In the next section, we deal with an arbitrary number $N$ of external legs, also starting in the negative-dimensional approach and showing how to obtain the Davydychev's original result calculated in the MB scheme; we carry out the same the integrals with FP. In the final section we present our conclusions and a discussion concerning the three methods.

## II. ONE-LOOP 2-POINT FUNCTION

In this section we present the calculations to evaluate the one-loop two-point scalar integral within NDIM scheme and compare this result with the obtained by Mellin-Barnes approach. Consider the integral,

$$I \;=\; \int d^Dk \exp\left\{-\alpha\left[k^2-m_1^2\right]-\beta\left[(k-q)^2-m_2^2\right]\right\}, \tag{2}$$

which is the usual generating functional for negative-dimensional integrals. We will always begin with this kind of generating functionals, for two and for $n$-point scalar integrals, and after some manipulations arrive at results which were obtained previously by other authors, using MB approach.

The first step in NDIM context is a series expansion of the above integral,

$$I \;=\; \sum_{a_1,a_2=0}^{\infty}\frac{(-\alpha)^{a_1}(-\beta)^{a_2}}{a_1!a_2!}J^{(2)}(a_1,a_2;q;m_1,m_2), \tag{3}$$

where we define the negative-dimensional integrals,

$$\begin{aligned} J^{(2)} \;&=\; J^{(2)}(a_1,a_2;q;m_1,m_2) \\ &=\; \int d^Dk\left[k^2-m_1^2\right]^{a_1}\left[(k-q)^2-m_2^2\right]^{a_2}. \end{aligned} \tag{4}$$

The integration (2) can be easily done,

$$I \;=\; \left(\frac{\pi}{\alpha+\beta}\right)^{D/2}\exp\left\{-\left(\frac{\alpha\beta}{\alpha+\beta}\right)q^2+\alpha m_1^2+\beta m_2^2\right\}, \tag{5}$$

and the exponential above expanded again in Taylor series,

$$I \;=\; \left(\frac{\pi}{\alpha+\beta}\right)^{D/2}\sum_{j_0=0}^{\infty}\frac{1}{j_0!}\left[-\left(\frac{\alpha\beta}{\alpha+\beta}\right)q^2+\alpha m_1^2+\beta m_2^2\right]^{j_0}.$$

Rewrite it as,

$$I \;=\; \pi^{D/2}\sum_{j_0=0}^{\infty}\frac{(\alpha+\beta)^{-D/2+j_0}\left(m_2^2\right)^{j_0}}{j_0!}\left[1-\frac{\alpha\beta}{(\alpha+\beta)^2}\frac{q^2}{m_2^2}-\frac{\alpha}{\alpha+\beta}\left(1-\frac{m_1^2}{m_2^2}\right)\right]^{j_0}, \tag{6}$$

and then a multinomial expansion gives us,

$$I = \pi^{D/2} \sum_{j_0,b_1,c_1=0}^{\infty} \frac{\alpha^{b_1+c_1}\beta^{b_1}\left(\alpha+\beta\right)^{-D/2+j_0-2b_1-c_1}\left(m_2^2\right)^{j_0}\Gamma(-j_0+b_1+c_1)}{\Gamma(1+j_0)\Gamma(-j_0)} \frac{\left(\frac{q^2}{m_2^2}\right)^{b_1}}{b_1!} \frac{\left(1-\frac{m_1^2}{m_2^2}\right)^{c_1}}{c_1!}, \tag{7}$$

that using

$$\left[\sum_{i=1}^{n}\alpha_i\right]^{-A} = \frac{1}{\alpha_1^A}\sum_{j_1,\ldots n-1=0}^{\infty} \frac{(A)_{j_1}}{(1)_{j_1}}\frac{(-j_1)_{j_2}}{(1)_{j_2}}\frac{(-j_2)_{j_3}}{(1)_{j_3}}\ldots\frac{(-j_{n-2})_{j_{n-1}}}{(1)_{j_{n-1}}}\left(-\frac{\alpha_2}{\alpha_1}\right)^{j_1}\left(-\frac{\alpha_3}{\alpha_2}\right)^{j_2}\left(-\frac{\alpha_4}{\alpha_3}\right)^{j_3}\ldots\left(-\frac{\alpha_n}{\alpha_{n-1}}\right)^{j_{n-1}}. \tag{8}$$

in the factor $(\alpha+\beta)^{-A}$ one obtains,

$$\begin{aligned} I &= \pi^{D/2}\sum_{j_0,j_1=0}^{\infty}\sum_{b_1,c_1=0}^{\infty}\alpha^{-D/2+j_0-b_1-j_1}\beta^{b_1+j_1}\frac{(-1)^{j_1}(m_2^2)^{j_0}\Gamma\left(-j_0+b_1+c_1\right)\Gamma\left(D/2-j_0+2b_1+c_1+j_1\right)}{\Gamma(1+j_0)\Gamma(-j_0)\Gamma\left(D/2-j_0+2b_1+c_1\right)\gamma(1+j_1)} \\ &\times\left(\frac{q^2}{m_2^2}\right)^{b_1}\left(1-\frac{m_1^2}{m_2^2}\right)^{c_1}\frac{1}{b_1!c_1!}, \end{aligned} \tag{9}$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$, is the Pochhammer symbol.

Compare the power of the parameters $\alpha$ and $\beta$ between (3) and (9) we have the following constraint equations,

$$a_1 = -D/2 + j_0 - b_1 - j_1 \tag{10}$$

$$a_2 = b_1 + j_1, \tag{11}$$

that after solving for $j_0$ and $j_1$ we have

$$j_0 = a_1 + a_2 + D/2 = \sigma_2 \tag{12}$$

$$j_1 = a_2 - b_1. \tag{13}$$

Performing the substitution of this result in (9) and the analytic continuation to $a_1, a_2 \leq 0$, we arrive at,

$$J^{(2)} = \pi^{D/2}(-m_2^2)^{\sigma_2}\frac{\Gamma(-\sigma_2)}{\Gamma(-a_1-a_2)}\sum_{b_1,c_1=0}^{\infty}\frac{(-\sigma_2)_{b_1+c_1}(-a_1)_{b_1+c_1}(-a_2)_{b_1}}{b_1!c_1!(-a_1-a_2)_{2b_1+c_1}}\left(\frac{q^2}{m_2^2}\right)^{b_1}\left(1-\frac{m_1^2}{m_2^2}\right)^{c_1}, \tag{14}$$

which is the well-known result for 2-point scalar integrals, with different masses, see [19].

### A. Two-point function via Feynman Parametrization

The most popular technique to deal with loop integrals is certainly, Feynman parametrization. It is the one the students learn on field theory courses, and one of the few textbooks introduce (the other is $\alpha-$parametrization). Depending on the manipulations one performs, it can turn the original loop integrals into a hefty one. We will proceed in a slightly different route. Our aim is to show how one can obtain the previous results for 2-point functions, given in terms of hypergeometric functions, using FP since in most cases the results calculated through FP are written as polylogarithms, $Li_n(z),\ n=0,1,2,3,4$.

Hypergeometric functions have an advantage over dilogarithms, for instance, in the case of photon-photon scattering scalar integrals. The result for $|s/4m^2|<1,\ |t/4m^2|<1$ can be written as a single Appel function $F_3$ of two variables and 5 parameters, on the other hand, the same result can be recast as a sum of several $Li_2(z_j)$ functions of complicated arguments $z_j$, see for instance[25].

Consider the function

$$\begin{aligned} F^{(2)} &= F^{(2)}(a_1,a_2;q;m_1;m_2;x_0,x_1) \\ &= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^{x_0}dx_1(x_0-x_1)^{a_1-1}(x_1-x_2)^{a_2-1}\int\frac{d^Dk}{\left\{\left[k^2-m_1^2\right](x_0-x_1)+\left[(k-q)^2-m_2^2\right](x_1-x_2)\right\}^{a_1+a_2}}. \end{aligned}$$

where $a_1, a_2 \geq 0$. We note that when $x_0 = 1,\ x_2 = 0$ we have the well-known Feynman parametrization to the propagator of $J^{(2)}$, that is $F^{(2)} = J^{(2)}$. Such modification will turn simpler the calculation of $N-$point integrals in section IIIA. This expression can be rewritten as follows,

$$\begin{aligned} F^{(2)} &= \frac{\Gamma(a_1+a_2)(x_0-x_2)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^{x_0} dx_1 (x_0-x_1)^{a_1-1}(x_1-x_2)^{a_2-1} \int d^D k \left\{ \left[ k - \frac{q(x_1-x_2)}{(x_0-x_2)^{1/2}} \right]^2 - m_2^2(x_0-x_2) \right. \\ &\quad \left. + \frac{q^2(x_0-x_1)(x_1-x_2)}{x_0-x_2} + (m_2^2-m_1^2)(x_0-x_1) \right\}^{-a_1-a_2}, \end{aligned}$$

that after the evaluation of the integral in $k$ using the well-known formula,

$$\int d^D k \frac{(k^2)^\alpha}{(k^2+M^2)^\beta} = \pi^{D/2} \left(M^2\right)^{\alpha+D/2-\beta} \frac{\Gamma(\beta-\alpha-D/2)\Gamma(\alpha+D/2)}{\Gamma(\beta)\Gamma(D/2)}. \tag{15}$$

we have

$$\begin{aligned} F^{(2)} &= \pi^{D/2} \frac{\Gamma(a_1+a_2-D/2)(x_0-x_2)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^{x_0} dx_1 (x_0-x_1)^{a_1-1}(x_1-x_2)^{a_2-1} \\ &\quad \times \left\{ -m_2^2(x_0-x_2) + \frac{q^2(x_0-x_1)(x_1-x_2)}{x_0-x_2} + (m_2^2-m_1^2)(x_0-x_1) \right\}^{D/2-a_1-a_2}, \end{aligned}$$

Taylor expanding, we get

$$\begin{aligned} F^{(2)} &= \pi^{D/2}(-m_2^2)^{D/2-a_1-a_2} \frac{(x_0-x_2)^{-a_1-a_2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{b_1=0}^{\infty} \sum_{c_1=0}^{\infty} (x_0-x_2)^{-b_1} x_0^{-b_1-c_1} \Gamma(a_1+a_2-D/2+b_1+c_1) \\ &\quad \times \frac{1}{b_1!} \left[ \frac{q^2}{m_2^2} \right]^{b_1} \frac{1}{c_1!} \left[ 1 - \frac{m_1^2}{m_2^2} \right]^{b_1} \int_0^{x_0} dx_1 (x_0-x_1)^{a_1+b_1+c_1-1}(x_1-x_2)^{a_2+b_1-1}. \end{aligned}$$

These integrals in $x_1$ can be evaluated, with $x_0 = 1,\ x_2 = 0$, using

$$\prod_{i=0}^{n-2} \left[ \int_0^{x_i} dx_{i+1} (x_i - x_{i+1})^{\alpha_{i+1}-1} \right] x_{n-1}^{\alpha_n - 1} = x_0^{\alpha_1+\ldots+\alpha_n-1} \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^n \alpha_i)}. \tag{16}$$

one obtains

$$\begin{aligned} F^{(2)} &= \pi^{D/2}(-m_2^2)^{D/2-a_1-a_2} \frac{\Gamma(a_1+a_2-D/2)}{\Gamma(a_1+a_2)} \sum_{b_1=0}^{\infty} \sum_{c_1=0}^{\infty} \frac{(a_1+a_2-D/2)_{b_1+c_1}(a_1)_{b_1+c_1}(a_2)_{b_1}}{b_1! c_1! (a_1+a_2)_{2b_1+c_1}} \left( \frac{q^2}{m_2^2} \right)^{b_1} \\ &\quad \times \left( 1 - \frac{m_1^2}{m_2^2} \right)^{c_1}, \end{aligned} \tag{17}$$

which is exactly the former result (14). This result show that, to one-loop two-point, the NDIM, Feynman parametrization and Mellin-Barnes representation are equivalent. The other kinematical regions can be obtained through analytic continuation of hypergeometric function above (see [19, 22, 26]).

## III. N-POINT FUNCTION

In this section we present the generalization of the previous ideas in order to the obtain the Mellin-Barnes result for the scalar integral associated to $n$-point function. We consider a one-loop Feynman diagram with $n$ external legs with momenta: $p_1 = l_2 - l_1, p_2 = l_3 - l_2, ..., p_n = l_1 - l_n$, and internal momenta $k - l_1, k - l_2, ..., k - l_n$. From a similar reasoning, we begin with the generating functional,

$$I_n = \int d^D k \exp\left\{ -\sum_{i=1}^n \alpha_i \left[ (k-l_i)^2 - m_i^2 \right] \right\} \tag{18}$$

$$
\begin{aligned}
&= \int d^D k \prod_{i=1}^{n} \sum_{a_i=0}^{\infty} \frac{(-\alpha_i)^{a_i}}{a_i!} \left[(k-l_i)^2 - m_i^2\right]^{a_i} \\
&= \sum_{a_1,\ldots,a_n=0}^{\infty} \frac{(-\alpha_1)^{a_1}}{a_1!} \ldots \frac{(-\alpha_n)^{a_n}}{a_n!} J^{(n)}\left(l_1, m_1, a_1; \ldots; l_n, , m_n, a_n\right), \qquad (19)
\end{aligned}
$$

where $J^{(n)}\left(l_1, m_1, a_1; \ldots; l_n, , m_n, a_n\right)$ represents the $n-$point functions for negative values of $a_i$ and is given by

$$
\begin{aligned}
J^{(n)} &= J^{(n)}\left(l_1, m_1, a_1; \ldots; l_n, , m_n, a_n\right) \\
&= \int d^D k \prod_{i=1}^{n} \left[(k-l_i)^2 - m_i^2\right]^{a_i}. \qquad (20)
\end{aligned}
$$

The expression (19), after the integration in $k$ can be rewritten of form

$$
I_n = \left(\frac{\pi}{\sum_{i=1}^n \alpha_i}\right)^{D/2} \exp\left\{-\frac{\sum_{i>j}^n \alpha_i \alpha_j l_{ij}^2}{\sum_{i=1}^n \alpha_i} + \sum_{i=1}^{n} \alpha_i m_i^2\right\},
$$

where $l_{ij} = l_i - l_j$. After a new expansion in the right side of the expression above, we have

$$
I_n = \left(\frac{\pi}{\sum_{i=1}^n \alpha_i}\right)^{D/2} \sum_{j_0=0}^{\infty} \frac{1}{\Gamma(1+j_0)} \left[-\frac{\sum_{i>j}^n \alpha_i \alpha_j l_{ij}^2}{\sum_{i=1}^n \alpha_i} + \sum_{i=1}^{n} \alpha_i m_i^2\right]^{j_0}, \qquad (21)
$$

using the expansions (1) and (8) for $[\sum_{i=1}^n \alpha_i]$ multinomial, with $N = n(n-1)/2$ terms, we get

$$
\begin{aligned}
I_n &= \pi^{D/2} \frac{1}{(2\pi i)^{N+n-1}} \sum_{j_0,\ldots,j_{n-1}=0}^{\infty} \frac{(-l_{12}^2)^{j_0}}{\Gamma(1+j_0)\Gamma(-j_0)} \int_{-i\infty}^{i\infty} \frac{(-1)^{j_1}}{\Gamma(1+j_1)} \prod_{i=2}^{n-1} \left[\frac{(-1)^{j_i}\Gamma(j_i - j_{i-1})}{\Gamma(1+j_i)\Gamma(-j_{i-1})}\right] \\
&\times \frac{\Gamma\left(\sum_{j>1,i<j}^{n} w_{ij} + \sum_{i=1}^n v_i - j_0\right) \Gamma(D/2 + j_0 + j_1 - \sum_{i=1}^n v_i)}{\Gamma\left(D/2 + j_0 - \sum_{i=1}^n v_i\right)} \prod_{j>2,i<j}^{n} \left[dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2}\right)^{w_{ij}} \Gamma(-w_{ij})\right] \\
&\times \prod_{i=1}^{n} \left[dv_i \left(-\frac{m_i^2}{l_{12}^2}\right)^{v_i} \Gamma(-v_i)\right] \prod_{i}^{n} \left[\alpha_i^{f_i}\right], \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &= -j_1 - D/2 - \sum_{i\neq 1, i<j}^{n} w_{ij} + v_1, \\
f_2 &= j_0 + j_1 - j_2 - \sum_{i\neq 2, i<j}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i, \\
f_i &= j_{i-1} - j_i + \sum_{j\neq i}^{n} w_{ij} + v_i, \quad i = 3, 4, \ldots, n-1, \\
f_n &= j_{n-1} + \sum_{j\neq n}^{n} w_{nj} + v_n. \qquad (23)
\end{aligned}
$$

We need to do now the comparison term-by-term between $\alpha_i$ powers in eq.(22) with the ones of eq.(19). We obtain $f_i = a_i$ and the solution of system above will be given by,

$$
\begin{aligned}
j_0 &= \sum_{i=1}^{n} a_i = \sigma_n, \\
j_1 &= -a_1 - D/2 - \sum_{i\neq 1, i<j}^{n} w_{ij} + v_1, \\
j_2 - j_1 &= -a_2 + \sigma_n - \sum_{i\neq 2, i<j}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i, \\
j_i - j_{i-1} &= -a_i + \sum_{j\neq i}^{n} w_{ij} + v_i, \quad i = 3, 4, ..., n-1 \\
j_{n-1} &= a_n - \sum_{j\neq n}^{n} w_{nj} - v_n.
\end{aligned} \tag{24}
$$

Performing the substitution of the solutions above in (22), we arrive at

$$
\begin{aligned}
J^{(n)} &= \pi^{D/2} (l_{12}^2)^{\sigma_n} \frac{(-1)^{a_1+...+a_n}(1)_{a_1+\epsilon}...(1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1}\left[\Gamma(1+\epsilon)\Gamma(-\epsilon)\right]^n} \int_{-i\infty}^{i\infty} \prod_{j>2, i<j}^{n} \left[ dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2}\right)^{w_{ij}} \Gamma(-w_{ij}) \right] \\
&\times \prod_{i=1}^{n} \left[ dv_i \left(-\frac{m_i^2}{l_{12}^2}\right)^{v_i} \Gamma(-v_i) \right] \prod_{i=3}^{n} \left[ \Gamma\left(-a_i + \sum_{j\neq i}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i\right) \right] \\
&\times \frac{\Gamma\left(\sum_{j>1, i<j}^{n} w_{ij} + \sum_{i=1}^{n} v_i - \sigma_n\right) \Gamma\left(\sigma_n - a_1 - \sum_{i\neq 1, i<j}^{n} w_{ij} - \sum_{i=2}^{n} v_i\right)}{\Gamma\left(D/2 + \sigma_n - \sum_{i=1}^{n} v_i\right)} \\
&\times \Gamma\left(-a_2 + \sigma_n - \sum_{i\neq 2, i<j}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i\right),
\end{aligned} \tag{25}
$$

(26)

that after carrying out analytic continuation to negative values of the $a_i$ provides,

$$
\begin{aligned}
J^{(n)} &= \pi^{D/2} (l_{12}^2)^{\sigma_n} \frac{1}{(2\pi i)^{N+n-1}} \prod_{i=1}^{n} \left[\frac{1}{\Gamma(-a_i)}\right] \\
&\times \int_{-i\infty}^{i\infty} \prod_{j>2, i<j}^{n} \left[ dw_{ij} \left(\frac{l_{ij}^2}{l_{12}^2}\right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^{n} \left[ dv_i \left(-\frac{m_i^2}{l_{12}^2}\right)^{v_i} \Gamma(-v_i) \right] \\
&\times \prod_{i=3}^{n} \left[ \Gamma\left(-a_i + \sum_{j\neq i}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i\right) \right] \Gamma\left(-a_2 + \sigma_n - \sum_{i\neq 2, i<j}^{n} w_{ij} - \sum_{i\neq 2}^{n} v_i\right) \\
&\times \frac{\Gamma\left(\sum_{j>1, i<j}^{n} w_{ij} + \sum_{i=1}^{n} v_i - \sigma_n\right) \Gamma\left(\sigma_n - a_1 - \sum_{i\neq 1, i<j}^{n} w_{ij} - \sum_{i=2}^{n} v_i\right)}{\Gamma(D/2 + \sigma_n - \sum_{i=1}^{n} v_i)}.
\end{aligned} \tag{27}
$$

The above result in also an expression for the $n$-point scalar integrals with arbitrary exponents of propagators and dimension, in the MB scheme. However, it was not this one the obtained by Davydychev in [23]. Formula (27) is a new result.

It is important to observe that the above result, eq.(27), is valid since the series is convergent which means $|l_{ij}^2/l_{12}^2| < 1$ and $|m_i^2/l_{12}^2| < 1$, that is, external momentum greater than masses. Conversely, Davydychev's result holds in

another kinematical region, namely, where $|l_{ij}^2/m_n^2| < 1$ and $|1 - m_i^2/m_n^2| < 1$, i.e., when masses are greater than incoming/outcoming momenta. This result could be obtained, in principle, from the Davydychev's formula through analytic continuation. However we stress the point that such analytic continuation formulas are not known for multiple hypergeometric series (in general these formulas are known only in the case of single and double series).

Other form to represent the n-point function can be obtained also from expansion of (21), that is

$$\begin{aligned} I_n &= \pi^{D/2}\frac{1}{(2\pi i)^{N+n-1}}\sum_{j_0,\ldots,n-1=0}^{\infty}\frac{(m_n^2)^{j_0}}{\Gamma(1+j_0)\Gamma(-j_0)}\int_{-i\infty}^{i\infty}\frac{(-1)^{j_1}}{\Gamma(1+j_1)}\prod_{i=2}^{n-1}\left[\frac{(-1)^{j_i}\Gamma(j_i-j_{i-1})}{\Gamma(1+j_i)\Gamma(-j_{i-1})}\right] \\ &\times\frac{\Gamma\left(\sum_{i<j}^{n}w_{ij}+\sum_{i=1}^{n-1}v_i-j_0\right)\Gamma(D/2+j_1+\sum\limits_{i,j\neq n,i<j}^{n}w_{ij})}{\Gamma\left(D/2+\sum\limits_{i,j\neq n,i<j}^{n}w_{ij}\right)} \\ &\times\prod_{i<j}^{n}\left[dw_{ij}\left(-\frac{l_{ij}^2}{m_n^2}\right)^{w_{ij}}\Gamma(-w_{ij})\right]\prod_{i=1}^{n-1}\left[dv_i\left(\frac{m_i^2}{m_n^2}\right)^{v_i}\Gamma(-v_i)\right]\prod_{i}^{n}[\alpha_i^{g_i}], \end{aligned} \tag{28}$$

where

$$\begin{aligned} g_i &= j_i - j_{i+1} + \sum_{j\neq i}^{n}w_{ij} + v_i, \quad i=1,2,...,n-2 \\ g_{n-1} &= j_{n-1} + \sum_{j\neq n-1}^{n}w_{n-1,j} + v_{n-1}, \\ g_n &= j_0 - j_1 - D/2 - 2\sum_{i,j\neq n,i<j}^{n}w_{ij} - \sum_{i=1}^{n-1}v_i, \end{aligned} \tag{29}$$

whose solution, after compare the $\alpha_i$ powers of the equation (28) with (19), $g_i = a_i$, is given by

$$\begin{aligned} j_0 &= \sum_{i=1}^{n}a_i = \sigma_n, \\ j_1 &= \sigma_n - D/2 - 2\sum_{i,j\neq n,i<j}^{n}w_{ij} - \sum_{i=1}^{n-1}v_i, \\ j_{i+1} - j_i &= -a_i + \sum_{j\neq i}^{n}w_{ij} + v_i, \quad i=1,2,...,n-2 \\ j_{n-1} &= a_{n-1} - \sum_{j\neq n-1}^{n}w_{n-1,j} - v_{n-1}. \end{aligned} \tag{30}$$

Analytically continuing and performing the substitution of the result above in (28) and compare its $\alpha_i$ powers with (19), we arrive at,

$$\begin{aligned} J^{(n)} &= \pi^{D/2}(-m_n^2)^{\sigma_n}\frac{(-1)^{a_1+\ldots+a_n}(1)_{a_1+\epsilon}\ldots(1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1}\left[\Gamma(1+\epsilon)\Gamma(-\epsilon)\right]^n}\int_{-i\infty}^{i\infty}\prod_{i=1}^{n-1}\left[\Gamma(-a_i+\sum_{j\neq i}^{n}w_{ij}+v_i)\right] \\ &\times\frac{\Gamma\left(\sum_{i<j}^{n}w_{ij}+\sum_{i=1}^{n-1}v_i-\sigma_n\right)\Gamma(\sigma_n-\sum\limits_{i,j\neq n,i<j}^{n}w_{ij}-\sum_{i=1}^{n-1}v_i)}{\Gamma\left(D/2+\sum\limits_{i,j\neq n,i<j}^{n}w_{ij}\right)} \end{aligned}$$

$$\times \prod_{i<j}^{n} \left[ dw_{ij} \left( -\frac{l_{ij}^2}{m_n^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \prod_{i=1}^{n-1} \left[ dv_i \left( \frac{m_i^2}{m_n^2} \right)^{v_i} \Gamma(-v_i) \right], \tag{31}$$

that can be rewritten of form

$$\begin{aligned} J^{(n)} &= \pi^{D/2}(-m_n^2)^{\sigma_n} \frac{(-1)^{a_1+...+a-n}(1)_{a_1+\epsilon}...(1)_{a_n+\epsilon}}{(2\pi i)^{N+n-1}\left[\Gamma(1+\epsilon)\Gamma(-\epsilon)\right]^n} \int_{-i\infty}^{i\infty} \prod_{i<j}^{n} \left[ dw_{ij} \left( -\frac{l_{ij}^2}{m_n^2} \right)^{w_{ij}} \Gamma(-w_{ij}) \right] \\ &\times \prod_{i=1}^{n-1} \left[ dv_i \left( \frac{m_i^2}{m_n^2} \right)^{v_i} \Gamma(-v_i)\Gamma(-a_i + \sum_{j\neq i}^{n} w_{ij} + v_i) \right] \\ &\times \frac{\Gamma\left(\sum_{i<j}^n w_{ij} + \sum_{i=1}^{n-1} v_i - \sigma_n\right)\Gamma(\sigma_n - \sum_{i,j\neq n, i<j}^{n} w_{ij} - \sum_{i=1}^{n-1} v_i)}{\Gamma\left(D/2 + \sum_{i,j\neq n,i<j}^{n} w_{ij}\right)}, \end{aligned} \tag{32}$$

that after the analytic continuation to negative values of $a_i$, we get

$$\begin{aligned} J^{(n)} &= \pi^{D/2}\left(-m_n^2\right)^{\sigma_n} \frac{\Gamma(-\sigma_n)}{\Gamma(-\sigma_n + D/2)} \sum_{b_{ij}=0}^{\infty} \sum_{c_l=0}^{\infty} \frac{(-\sigma_n)_{\sum_{i>j} b_{ij} + \sum_l c_l} \prod_{i=1}^{n-1}\left[(-a_i)_{\sum_{i\neq j} b_{ij}+c_i}\right](-a_n)_{\sum_{j\neq n} b_{nj}}}{(\sigma_n + D/2)_{2\sum_{i>j} b_{ij} + \sum_l c_l}} \\ &\times \prod_{i>j}^{n} \left[ \frac{1}{b_{ij}!} \left( \frac{l_{ij}^2}{m_n^2} \right)^{b_{ij}} \right] \prod_{l=1}^{n-1} \left[ \frac{1}{c_l!} \left( 1 - \frac{m_i^2}{m_n^2} \right)^{c_l} \right] \end{aligned} \tag{33}$$

which was the result obtained by Davydychev in [23] using MB approach. So, with the generating functional (18) as the starting point — the same which we have, in a previous paper [27] in the NDIM approach, used to show how to obtain a general formula to *any* scalar one-loop Feynman integrals, in covariant gauges — we were able to reproduce a MB result eq.(33) and more, to present another formula (27), also valid for $n$-point scalar integrals.

### A. N-Point function via Feynman Parametrization

Our final task in this paper is to show how to solve an $N$-point scalar integral using FP technique. As far as we know there is no such result in the literature calculated using FP. Of course, it has to be the same we obtained before using NDIM and Davydychev's [23] MB approaches.

We start with the function $F^{(n)}$

$$\begin{aligned} F^{(n)} &= F^{(n)}\left(a_i; l_i; m_i; x_0, x_n\right) \\ &= \frac{\Gamma\left(a_1 + ... + a_n\right)}{\Gamma\left(a_1\right)...\Gamma\left(a_n\right)} \prod_{i=0}^{n-2} \left[ \int_0^{x_i} dx_i \left(x_i - x_{i+1}\right)^{a_{i+1}-1} \right] \left(x_{n-1} - x_n\right)^{a_n - 1} \\ &\times \int \frac{d^D k}{\left\{ \prod_{i=1}^n \left[ \left(k - l_i\right)^2 - m_i^2 \right] \left(x_{i-1} - x_i\right) \right\}^{\sum_{i=1}^n a_i}}, \end{aligned} \tag{34}$$

where $a_i \geq 0$. This function for $x_0 = 1$ $x_n = 0$, represent the Feynman parametrization to the integral of type (20). The integral in $k$ above can be evaluated using (15),

$$\begin{aligned} F^{(n)} &= \pi^{D/2} \frac{\Gamma(a_1 + ... + a_n - D/2)\left(x_0 - x_n\right)^{-D/2}}{\Gamma\left(a_1\right)...\Gamma\left(a_n\right)} \prod_{i=0}^{n-2} \left[ \int_0^{x_i} dx_i \left(x_i - x_{i+1}\right)^{a_{i+1}-1} \right] \left(x_{n-1} - x_n\right)^{a_n - 1} \\ &\times \left\{ \sum_{i=1}^{n} \left(l_i^2 - m_i^2\right)\left(x_{i-1} - x_i\right) - \frac{1}{x_0 - x_n} \left[ \sum_{i=1}^{n} l_i (x_{i-1} - x_i) \right]^2 \right\}^{\sum_i a_i - D/2}. \end{aligned} \tag{35}$$

Using now

$$\sum_{i=1}^{n} l_i^2 \left(x_{i-1}-x_i\right) - \frac{\left[\sum_{i=1}^{n} l_i (x_{i-1}-x_i)\right]^2}{x_0 - x_n} = \sum_{i>j}^{n} \frac{(x_{i-1}-x_i)(x_{j-1}-x_j)}{x_0-x_n} \left(\frac{l_{ij}^2}{m_n^2}\right) \tag{36}$$

and

$$\sum_{i=1}^{n-1} m_i^2 \left(x_i - x_{i-1}\right) = -m_n^2 (x_0 - x_n) + \sum_{i=1}^{n-1} \left(m_i^2 - m_n^2\right)\left(x_i - x_{i-1}\right). \tag{37}$$

performing also the Taylor expansion of the argument of the integral above, we get

$$\begin{aligned} F^{(n)} \;&=\; \pi^{D/2}\left(-m_n^2\right)^{D/2-\sum_i a_i} \frac{1}{\Gamma\left(a_1\right)...\Gamma\left(a_n\right)} \sum_{b_{ij}=0}^{\infty}\sum_{c_l=0}^{\infty} \left(x_0-x_n\right)^{-D/2-2\sum_{i>j}^{n} b_{ij} - \sum_{l=1}^{n} c_l} \\ &\times \Gamma\left(\sum a_i - D/2 + \sum_{i>j} b_{ij} + \sum_{l=1}^{n-1} c_l\right) \prod_{i>j}^{n}\left[\frac{1}{b_{ij}!}\left(\frac{l_{ij}^2}{m_n^2}\right)^{b_{ij}}\right] \prod_{l=1}^{n-1}\left[\frac{1}{c_l!}\left(1-\frac{m_i^2}{m_n^2}\right)^{c_l}\right] \\ &\times \prod_{i=0}^{n-2}\left[\int_0^{x_i} dx_i \left(x_i - x_{i+1}\right)^{g_{i+1}-1}\right]\left(x_{n-1}-x_n\right)^{g_n -1}, \end{aligned} \tag{38}$$

where

$$g_i \;=\; a_i + \sum_{j\neq i} b_{ij} + c_i, \quad , i=1,2,...,n-1 \tag{39}$$

$$g_n \;=\; a_n + \sum_{j\neq n} b_{nj}. \tag{40}$$

The integral above can be evaluated with help of (16). If we take $x_0 = 1,\ x_n = 0,\ F^{(n)} = J^{(n)}$, we arrive

$$\begin{aligned} J^{(n)} \;&=\; \pi^{D/2}\left(-m_n^2\right)^{D/2-\sum_i a_i} \frac{\Gamma\left(\sum a_i - D/2\right)}{\Gamma\left(\sum a_i\right)} \\ &\times \sum_{b_{ij}=0}^{\infty}\sum_{c_l=0}^{\infty} \frac{\left(\sum a_i - D/2\right)_{\sum_{i>j} b_{ij} + \sum_l c_l} \prod_{i=1}^{n-1}\left[(a_i)_{\sum_{i\neq j} b_{ij} + c_i}\right](a_n)_{\sum_{j\neq n} b_{nj}}}{\left(\sigma_n + D/2\right)_{2\sum_{i>j} b_{ij} + \sum_l c_l}} \prod_{i>j}^{n}\left[\frac{1}{b_{ij}!}\left(\frac{l_{ij}^2}{m_n^2}\right)^{b_{ij}}\right] \\ &\times \prod_{l=1}^{n-1}\left[\frac{1}{c_l!}\left(1-\frac{m_i^2}{m_n^2}\right)^{c_l}\right]. \end{aligned} \tag{41}$$

This result is the same one obtained in the previous subsection via NDIM in (33). This agreement show that, in one-loop level, the NDIM, Feynman parametrization and Mellin-Barnes representation present the same results and are equivalent: all of them can be used to solve all scalar Feynman loop integrals at one-loop order, with general masses, arbitrary exponents of propagators and dimension.

## IV. DISCUSSION AND CONCLUSION

So far we have made calculations in order to show that the same class of generating functionals can be used to reproduce MB results. Depending on which Taylor expansions one carries out one can proceed in the NDIM or MB routes. The final results will be, obviously, the same, given in terms of generalized hypergeometric functions, being the exponents of propagators and space-time dimension arbitraries.

However, one could ask which of these two routes, if any, is the one where Feynman integrals become simpler to solve. The first point to observe what are the tools one has to master in order to tackle such integrals in both approaches: contour integration, Cauchy theorem and Barnes' lemmas for MB, and solving system of algebraic equations for

NDIM. So far, so good. Second, the results, despite they will be the same, have to be worked out one-by-one in the MB context, on the other hand, using NDIM and solving the system of algebraic equations gives one all the possible solutions (generalized hypergeometric functions) for the Feynman integral in question. Group them is a straightforward task: linear independent functions have to be summed, each set is a possible result in a given region if convergence[26]. Third, the massless case needs to be known in the case of MB in order to tackle massive integrals; not so in NDIM.

We can summarize both approaches in following table,

| Step | $MB$ | $NDIM$ |
|---|---|---|
| 1 | Generating functional | Generating functional |
| 2 | Solve it | Solve it |
| 3 | Taylor expand (whole) | Taylor expand (each or whole) |
| 4 | Mellin transform | Project powers |
| 5 | Compare term-by-term | Compare term-by-term |
| 6 | Solve it for the integral | Solve it for the integral |
| 7 | Result: parametric integrals | Result: system of algebraic equations |
| 8 | Choose the contour: left or right | Elementary techniques |
| 9 | Cauchy theorem | Use the results of the systems |
| 10 | and Barnes' lemmas | Analytically continue to positive $D$ |
| 11 | One have one final result among several | One have all the series (final results) |

in the step number 3 one can proceed as we have done in this paper, expanding the exponential, or as we did in our previous works taking a Taylor expansion for each argument of the exponential. The final step, 11, is to be understood in the following manner: in order to obtain all possible generalized hypergeometric functions (which come in NDIM) using MB one has to repeat the above procedure choosing other sequence of contours, we mean for instance left-left-right-left-right and another one left-left-right-right-right, these two can give, in principle distinct generalized hypergeometric series. Some of them, will of course result in zero, since there can be no poles inside the contours. These ones are also contained in the NDIM approach, since some determinants can vanish, a much simpler calculation that can be implemented in softwares like Mathematica.

The textbook technique, FP, can be made simpler if one introduces two extra parameters $x_0$ and $x_n$, and takes series expansions in the parameters $(x_0 - x_1),\ (x_1 - x_2),\ ..., (x_{n-1} - x_n)$. In the end of the day one makes $x_0 = 0$, $x_n = 1$ and uses the well-known beta function integral representation. Then, the remaining expression is the result written as a generalized hypergeometric function.

### A. Conclusion

We have shown that negative-dimensional integration method (NDIM), Feynman parametrization (FP) and Mellin-Barnes' approach to scalar Feynman loop integrals, at one-loop level, give the same results. It depends only on how one choose to Taylor expand the generating functional (18). We present detailed calculations for two-point scalar integrals, with arbitrary masses, exponents of propagators and space-time dimension (in covariant gauges). Then we tackle a general scalar $N$-point integral, with different masses, and did show that the general formula of Davydychev [23] and ours [27] agree, as well as another one obtained via FP worked out, as far as we know, for the first time. It is our opinion however, that NDIM is simpler than MB, since all the possible results for the integral in question are obtained simultaneously, and in MB they must be calculated one by one, or through analytic continuation formulas, if such formulas were known, depending on the hypergeometric functions. FP is also a very powerful technique if one introduces two extra parameters and take Taylor expansions properly. In doing so, FP can become even simpler than NDIM, since one obtain the full result and does not have the drawback of searching among a huge amount of possible solutions.

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